Taming Irrationality: An Invariance Principle for the Random Billiard Walk

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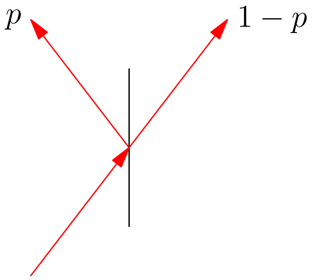
The Random Billiard Walk

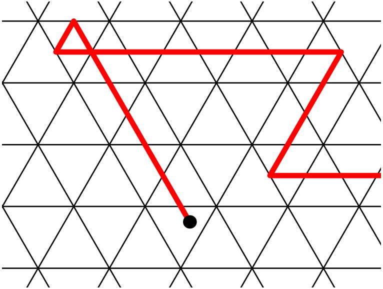
We study a random billiard (or laser) in an infinite equilateral triangle grid.

Definition (Defant – Jiradilok – Mossel, 2025+)

The **random billiard walk** on W is the trajectory $(L_t)_{t \geq 0}$ of a laser which

- starts at $L_0 \in \mathbb{R}^2$ with direction $u \in S^1$ with constant velocity,
- reflects with probability $p \in (0, 1)$ every time it hits a line.





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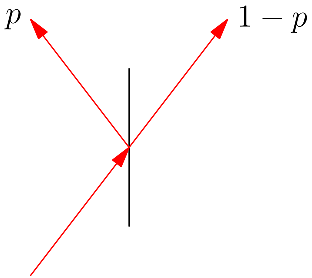
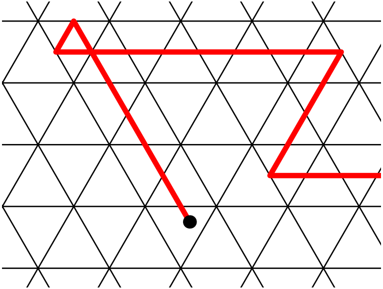
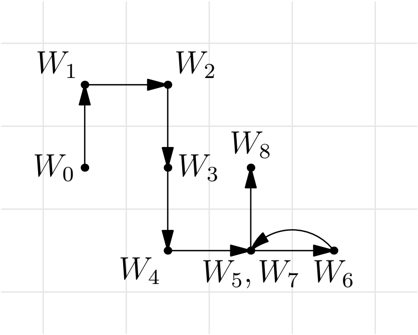




Question: What can we say about the distribution of L_t, as $t \to \infty$?

A Standard Random Walk on $\mathbb{Z}^2 \subset \mathbb{R}^2$

Easier Question: Consider a random walk $(W_n)_{n \in \mathbb{N}}$ on $\mathbb{Z}^2$ by $W_{n+1} = W_n + \varepsilon_n$, with i.i.d. steps $\varepsilon_n \sim \text{Unif}\{(0, \pm 1), (\pm 1, 0)\}$.



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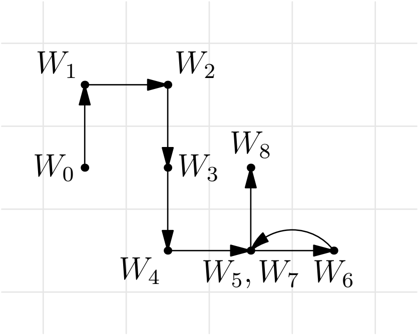


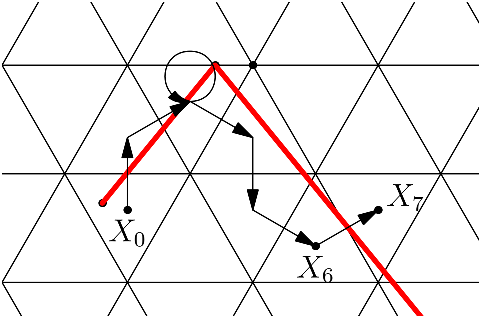
What can we say about the distribution of W_n, as $n \to \infty$?

Answer: By the Central Limit Theorem

$$\frac{W_n}{\sqrt{n}} = \frac{W_0 + \varepsilon_1 + \cdots + \varepsilon_n}{\sqrt{n}} \xrightarrow{d} \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}\right)$$

A Standard Random Walk on $\mathbb{Z}^2 \subset \mathbb{R}^2$

We could try applying this framework to our problem. Discretize $(L_t)_{t\in\mathbb{R}}$ to $(X_n)_{n\in\mathbb{N}}$ by recording the triangle the laser is in after every step.



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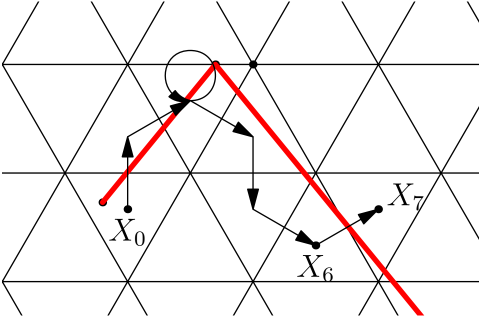

Critically, the i.i.d. assumption on $\Delta X_n = X_{n+1} - X_n$ fails (in both ways):

1. $(\Delta X_n)_{n\in\mathbb{N}}$ are not independent: the walk has "momentum".
2. $(\Delta X_n)_{n\in\mathbb{N}}$ are not identically distributed: for a general direction, the laser "cuts" the grid differently at different times.

Main Results

By restricting the direction we can remove the first obstruction.

Definition

A direction is **rational** if it lies in the lattice of the triangular grid.

Theorem (Defant – Jiradilok – Mossel, 2025)

Fix $p \in (0,1)$ *and* $L_0 \in \mathbb{R}^r$*. If the initial direction is rational, then*

$$\frac{L_t}{\sqrt{t}} \xrightarrow[t\to\infty]{d} \mathcal{N}(0, \sigma^2 I_r)$$

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Using a new approach, we remove this assumption.

Theorem (C., 2025+)

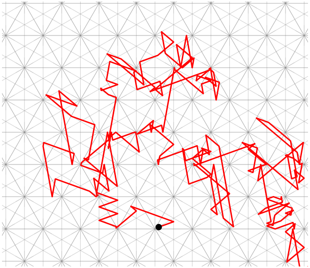
This conclusion holds for every initial direction.

Main Results

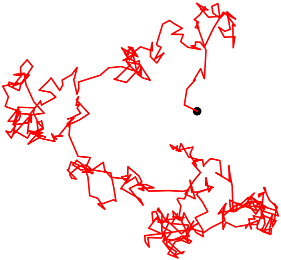
More strongly, we prove an invariance principle about the *entire trajectory*.

Theorem (C., 2025+)

The sequence of random normalized trajectories $F_n : [0,1] \to \mathbb{R}^d$, defined by $F_n(t) = L_{tn}/\sqrt{n}$, converges in distribution to Brownian motion.



(a) Brownian motion in $\mathbb{R}^2$

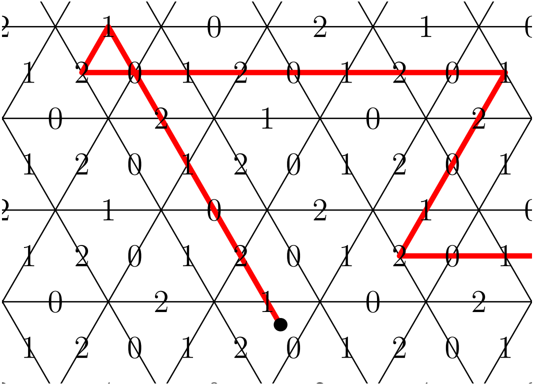


(b) $L_{tn}/\sqrt{n}$ in $\widetilde{A}_2$

Proof Overview

A key idea is that the Random Billiard Walk is both:

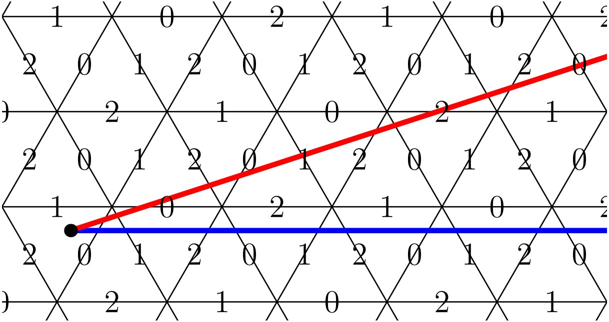
- A continuous time stochastic process in $\mathbb{R}^r$.
- A random walk on the group W generated by all reflections.



Group elements correspond to triangles. Edges can be numbered with $i \in \{0, 1, 2\}$ s.t. crossing edge i corresponds to multiplying by $s_i \in W$.

Proof Overview

Observe that this labeling is invariant under reflection! Therefore, X_n lies in the triangle corresponding to $w_n := s_{i_1}^{\varepsilon_1} s_{i_2}^{\varepsilon_2} \cdots s_{i_n}^{\varepsilon_n}$, $\varepsilon_i \sim \mathrm{Ber}(1-p)$, where $I = (i_1, i_2, \dots)$ is the **cutting sequence** of the initial direction.



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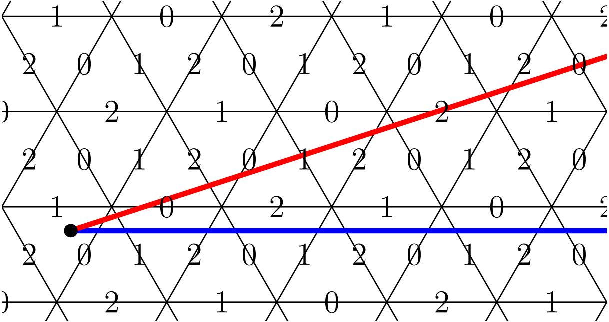

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The direction u is rational if and only if I is periodic, so we can then group $(L_t)_{t\in\mathbb{R}}$ into parts that behave the same way. The general case is harder.

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We wish to show that, qualitatively, the random billiard walk has similar properties to the i.i.d. random walk on $\mathbb{Z}^2$. This has two parts.

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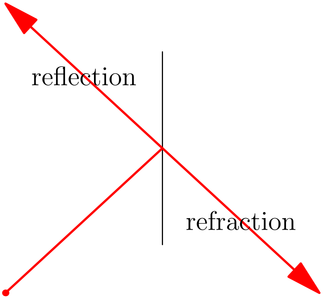
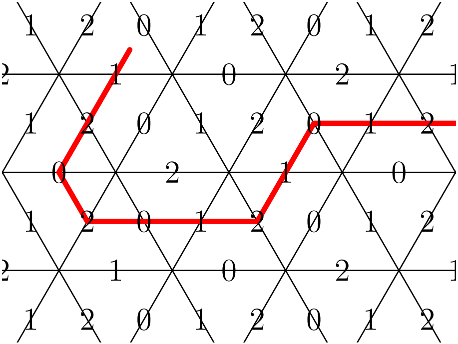
Thus, we can construct a **martingale** $(M_n)_{n\in\mathbb{N}}$ that closely tracks $(X_n)_{n\in\mathbb{N}}$, and thus $(L_t)_{t\geq 0}$. We can then apply standard convergence results.

Further directions

Question: What if we replaced reflections with refractions?





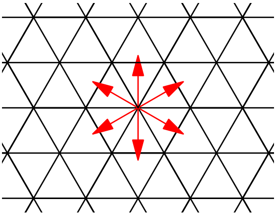
This defines the **Random Refraction Billiard Walk** $(R_t)_{t\geq 0}$.

Conjecture (C., 2025+)

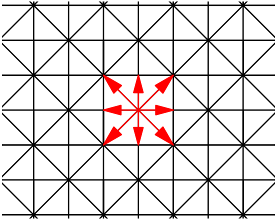
For every p, R_0, u, the trajectories $R_{tn}/\sqrt{n}$ converge to Brownian motion.

Further Directions

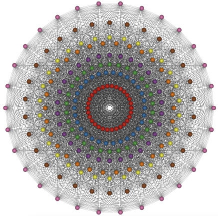
So far, all our results work for hyperplane arrangements of any irreducible affine Weyl group ($\widetilde{A}_n$, $\widetilde{B}_n$, $\widetilde{D}_n$, $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$, $\widetilde{F}_4$, $\widetilde{G}_2$).



(a) $\widetilde{A}_2$



(b) $\widetilde{B}_2$



(c) The $\widetilde{E}_8$ root system

Theorem (C., 2025+)

The refraction conjecture holds for $W = \widetilde{A}_n$, $\widetilde{G}_2$ *and* $\widetilde{B}_2$.

Can we make the proof type-uniform?

Acknowledgements

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